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ADVANCES IN THE PARTICLE FINITE ELEMENT METHOD FOR FSI: A MODIFIED FRACTIONAL STEP APPROACH

PAVEL RYZHAKOV*, EUGENIO OÑATE*, RICCARDO ROSSI* AND SERGIO IDELSOHN†

*Centre Internacional de Mètodes Numèrics en Enginyeria (CIMNE)
Universitat Politécnica de Catalunya
Campus Norte UPC, 08034 Barcelona, Spain
e-mail: pryzhakov@cimne.upc.edu, www.cimne.com/

†ICREA Research Professor at CIMNE

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Abstract. The Particle Finite Element Method was originally developed for applications involving incompressible fluids containing free surfaces. The PFEM's fundamental features include a) updated Lagrangian description b) nodal storage of variables, c) remeshing, d) alpha-shape based free surface determination. Several fluid formulations sharing these basic ideas of the PFEM [1] have been developed over the past decade. One can classify the existing PFEM fluid formulations into 2 main groups: purely incompressible and quasi-incompressible ones. It was found that the quasi-incompressible formulations led to a number of advantages when applied to FSI problems involving flexible structures. However, in these latter formulations, the desired incompressible solution was only approximated and numerical difficulties when approaching the incompressibility limit (i.e. using very large values of the compressibility constant) were faced.

In the present work we introduce the PFEM formulation of the new generation: it combines the best features of the previously developed methods and leads to incompressible solution, while preserving some features of the quasi-incompressible formulations advantageous for the FSI.

1 INTRODUCTION

The Particle Finite Element Method (PFEM) was orginally developed for solving problems involving gravitational incompressible flows containing free surfaces and was found to be advantageous for the fluid-structure interaction (FSI) [1]. Several fluid formulations/strategies have been utilized within the overall framework of the PFEM over the past years.

Fractional step method is possibly the most widely used approach for solving the incompressible Navier-Stokes equations due to the efficiency gained by decoupling velocity and pressure variables. The first works on the Particle Finite Element Method (PFEM) utilized pressure segregation via fractional step [1]. On the other hand, penalty and related quasi-incompressible methods define another competetive option and lead to certain advantages. In particular, it was shown that for the solution of fluid-structure interaction (FSI) problems, introducing the slight compressibility into the fluid leads to significant advantages for the FSI[3], [2]. One may naturally define a monolithic FSI solver, permitting to treat cases (such as interaction with light-weight structure) where many other approaches fail. While leading to efficient FSI strategies, a certian number of drawbacks was associated to the quasi-incompressibile methods: a) poor pressure stability for high values of the compressibility constant (incompressible limit) b) the solution only approximates the true incompressible solution.

In the present work we propose a methodology that combines the ideas of the previous works. We strive to develop a truely incompressible formulation, while preserving the features of quasi-incompressible approaches advantageous for coupling with the structure in FSI. This is achieved by modifying the pressure term in the fractional momentum equation. The idea is to use a prediction upon the end-of-step pressure. The objective is to use such an approximation that should not involve resolution of any equations system, thus not increasing the computational effort considerably. Including the approximated end-of-step pressure (instead of the previous step pressure) in the fractional momentum equation results in an intermediate velocity being much closer to the end-of-step velocity, than in a standard fractional step procedure. Thus, the convection can be resolved much better.

As in the classical fractional step method, once the fractional velocity is computed, the Poisson's equation is solved for the pressure. The Poisson' equation in our technique serves as a corrector acting upon the predicted pressure field to give the final incompressible end-of-step pressure. Having computed the end-of-step pressure, the correction step is applied in a standard manner and the incompressible velocity end-of-step velocity is obtained.

2 MODIFIED FRACTIONAL STEP

The discrete momentum-continuity system (for simplicity we use BE scheme for the illustration; the details upon the space discretization are omitted) for the incompressible flow in the updated Lagrangian framework can be written as:

$$\mathbf{r}_m = \mathbf{F} - \left(\rho \mathbf{M} \frac{\bar{\mathbf{v}}_{n+1} - \bar{\mathbf{v}}_n}{\Delta t} + \mu \mathbf{L} \bar{\mathbf{v}}_{n+1} + \mathbf{G} \bar{p}_{n+1} \right) \tag{1}$$

$$\mathbf{D} \bar{\mathbf{v}}_{n+1} = 0 \tag{2}$$

where $\mathbf{r}_m$ is the momentum equation residual[1], $\mathbf{M}$ is the mass matrix, $\mathbf{L}$ is the Laplacian matrix, $\mathbf{G}$ is the gradient matrix, $\bar{\mathbf{v}}$, $\bar{p}$ and μ are the velocity, pressure and viscosity respectively and $\mathbf{F}$ is the body force vector.

Next we introduce the modified fractional step split, that consists in using a pressure prediction in the fractional momentum equation:

$$\tilde{\mathbf{r}}_m = \mathbf{F} - \left(\rho\mathbf{M}\frac{\tilde{\mathbf{v}} - \bar{\mathbf{v}}_n}{\Delta t} + \mu\mathbf{L}\tilde{\mathbf{v}} + \mathbf{G}\bar{p}^g_{n+1}\right) \tag{3}$$

$$\rho\mathbf{M}\frac{\bar{\mathbf{v}}_{n+1} - \tilde{\mathbf{v}}}{\Delta t} + \mathbf{G}\left(\bar{p}_{n+1} - \bar{p}^g_{n+1}\right) = 0 \tag{4}$$

$$\mathbf{D}\bar{\mathbf{v}}_{n+1} = 0 \tag{5}$$

The pressure Poisson equation is obtained by applying the incompressibility condition 5 to the end-of-step momentum equation, leading to

$$\mathbf{D}\tilde{\mathbf{v}} = \Delta t\mathbf{D}\mathbf{M}^{-1}\mathbf{G}\left(\bar{p}_{n+1} - \bar{p}^g_{n+1}\right) \tag{6}$$

For simplicial equal interpolation order mixed elements, pressure needs to be stabilized. This consists in adding a stabilization term into the pressure equation. Indicating the stabilization term (we do not specify here the particular stabilization technique) as $\tau\mathbf{S}$ and using the approximation $\mathbf{D}\mathbf{M}^{-1}\mathbf{G} \approx \mathbf{L}$, we arrive at the final system:

$$\tilde{\mathbf{r}}_{mom} = \mathbf{F} - \left(\mathbf{M}\frac{\tilde{\mathbf{v}} - \bar{\mathbf{v}}_n}{\Delta t} + \mu\mathbf{L}\tilde{\mathbf{v}} + \mathbf{G}\bar{p}^g_{n+1}\right) \tag{7}$$

$$\mathbf{D}\tilde{\mathbf{v}} = \Delta t\mathbf{L}\left(\bar{p}_{n+1} - \bar{p}^g_{n+1}\right) + \tau\mathbf{S}\bar{p}_{n+1} \tag{8}$$

$$\mathbf{M}\frac{\bar{\mathbf{v}}_{n+1} - \tilde{\mathbf{v}}}{\Delta t} + \mathbf{G}\left(\bar{p}_{n+1} - \bar{p}^g_{n+1}\right) = 0 \tag{9}$$

Fractional momentum equation solution To solve the fractional momentum equation, the pressure prediction $\bar{p}^g$ shall be computed. Note that assuming that it is equal to zero ($\bar{p}^g = 0$) or to the previous step pressure ($\bar{p}^g = \bar{p}_n$) would recover standard first and second order fractional step schemes respectively. In this work we propose to obtain the prediction by associating the pressure increment with the volume change as (this way of computing pressure is presented in detail in [3]), equivalent to assuming the slight compressibility.

$$\delta\bar{p}^I = \kappa\frac{\delta V^I}{V^I_n} \tag{10}$$

where $\bar{p}^I$ is the pressure at the node I, κ is the compressibility constant and V^I_n is the nodal volume (we associate to each node of the FE mesh a nodal volume defined in such

[1] Note that writing the equations in the updated Lagrangian framework, i.e. in the unknown configuration $\mathbf{x}_{n+1}$ makes the discrete operators non-linear and obliges us to use residual form.

a way that it coincides with the diagonal entry of the diagonalized pressure mass matrix $\bar{V}_I := \mathbf{M}_{p,II}$).

Application of Newton's method to the solution of the non-linear fractional momentum equation (Eq. 7) requires to evaluate the dynamic tangent matrix $\mathbf{H}$:

$$\mathbf{H} = \frac{\partial \mathbf{r}_m^{frac}}{\partial \mathbf{v}} = \frac{\mathbf{M}}{\Delta t^2} + \mu \mathbf{L} + \kappa \mathbf{G}\mathbf{M}^{-1}\mathbf{D} \tag{11}$$

where the last term corresponds to the linearization of the pressure gradient (see [3]).

One can see, that the tangent of the fractional momentum equation contains the volumetric term, i.e. linearization of the pressure gradient. The additional computational cost due to the pressure update is minimal, as it does not involve solution of any system, provided that the unknown pressure is multiplied by the lumped pressure mass matrix. The cost of adding the $\kappa \mathbf{G}\mathbf{M}^{-1}\mathbf{D}$ term to the tangent matrix can be minimized by using a matrix-free method (see [3]).

Pressure Poisson' equation and the correction step The next step to be carried out is the correction of the pressure, i.e. obtaining the end-of-step incompressible pressure. This is done by solving Eq. 8. Solution of Eq. 8 requires to impose the pressure boundary condition. While usually zero pressure is fixed at the free surface, we propose to use a physically more meaningful option, i.e. the predicted pressure. The predicted "quasi-incompressible" pressure is a meaningful physical approximation, provided that the compressibility constant κ used in the pressure prediction is large.

This step can be thus viewed as a correction of the predicted pressure $\bar{p}^g_{n+1}$ to the correct end-of-step one everywhere except for the free surface, where the predicted pressure is kept as a Dirichlet b.c..

The correction step is carried out according to Eq. 9 and returns the end-of-step divergence-free velocity.

3 FLUID-STRUCTURE INTERACTION

The method proposed can be easily implemented within monolithic FSI strategies proposed in our previous works [3], [2]. According to these approaches a unique discretization is applied to the entire domain, containing the fluid and the structure where a single monolithic FSI system of equations is solved. Thus the interaction becomes an intrinsic feature of the method and does not involve coupling iterations and boundary condition exchange between the fluid and the structure sub-domains. The fractional momentum equation for the fluid and the momentum equation of the structure are assembled into a single system of equations. This step completely coincides with the procedure proposed in [3], [2]. However, in former strategies the obtained velocity field was considered to be the end-of-step velocity, thus approximating the incompressible behavior by the slightly compressible one. In the present approach two further steps are carried out. The first one is the solution of the Poisson's equation in the fluid domain. This allows the use conventional pressure

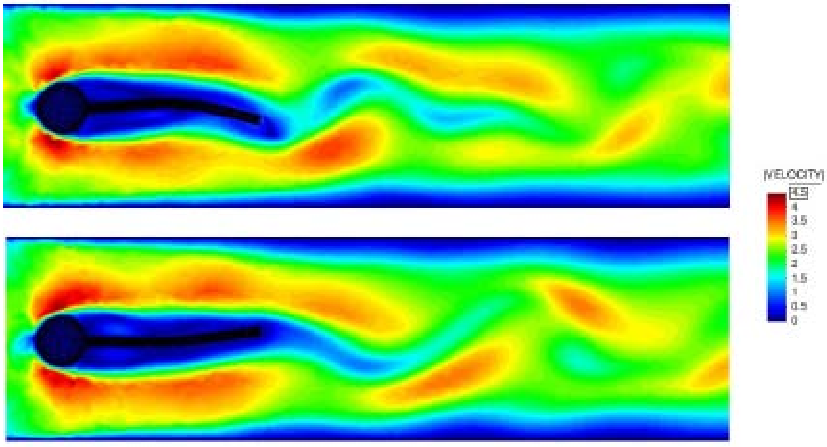

Figure 1: Velocity field and the deformed structure at two time instances.

stabilization techniques and smooths the pressure field, which is particularly important when high values of the compressibility constant is used at the prediction step. This step affects exclusively the fluid domain. In order to maintain the strong coupling between the fluid and the structure it is important to modify the correction step. In order to obtain the end-of-step velocity, instead of the projection step (Eq. (9)) carried out exclusively for the fluid domain, we propose to re-solve the monolithic FSI momentum equation using the gradient of the newly obtained end-of-step pressure. The approach can be also applied in the partitioned context. There is some evidence that due to the presence of the predicted pressure in the fractional momentum equation, classical Dirichlet-Neumann (failing when standard fractional step approach is used) coupling may work.

4 EXAMPLE

This example tests the performance of the method in its application to a fluid-structure interaction problem. The benchmark is described in [4] and deals with a laminar flow over a fixed cylinder, with an elastic beam attached to it. The velocity field and horizontal and vertical displacements of the beam are analyzed. The chosen material properties correspond to the most challenging FSI setting: density ratio between the fluid and the structure is equal to 1.

Fig. 1 shows the velocity field at two time instances and Fig. 2 displays the comparison of the temporal evolution of vertical and horizontal displacements of point A (comparison with the results of [4]).

The results obtained in both test cases and their comparison with the results presented in [4] prove applicability of the method in the field of the FSI. Numerical tests have shown that already on coarse meshes the behavior of the flow and its interaction with the structure can be well-reproduced. In comparison with the classical fractional step method, the approach proposed here did not require any additional techniques in order

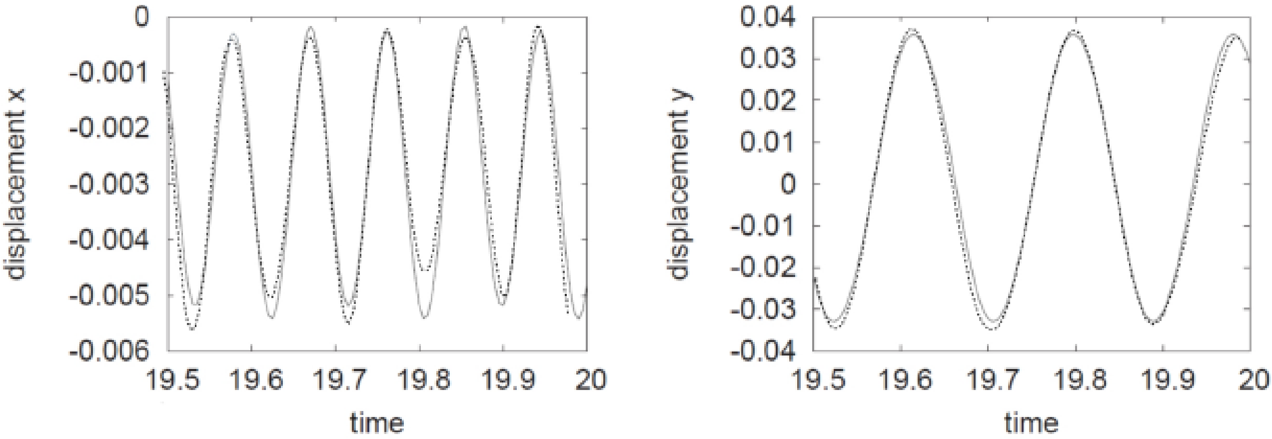

Figure 2: Displacements of the beam tip. Dashed line: present method, continuous line: Turek [4]. TC2

to ensure the convergence of the coupled FSI problem.

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